Formal Meta-Theory of Bisimulation Up To in Abella

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MSC Week 5: Concurrency, Logic, and Types

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Outline

- **Formalizing Bisimulation in Abella**
- **Bisimulation Up To**
- **Major Up To Techniques (for CCS)**
- **Case of the π-Calculus**
- **Perspectives**

Bisimulation in Abella

`http://abella-prover.org`

Formally: Progress

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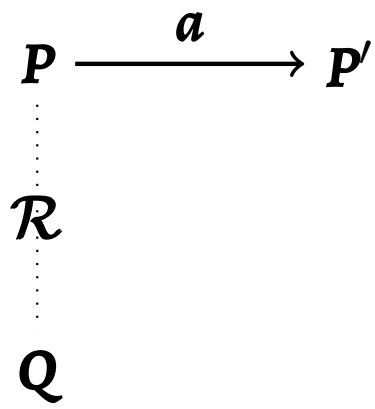

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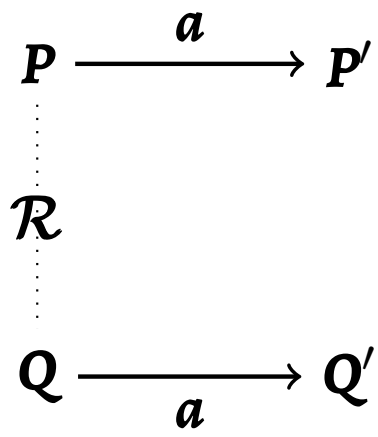

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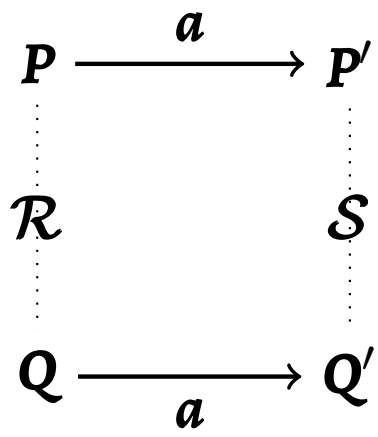

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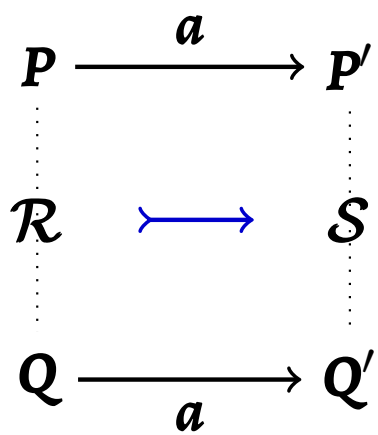

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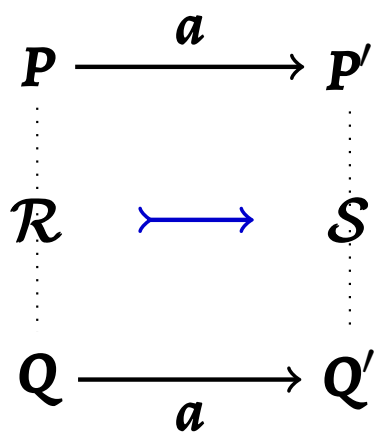

$$\mathcal{R} \rightarrowtail \mathcal{S} \triangleq$$
$$\forall P, Q.\ P\,\mathcal{R}\,Q \implies$$
$$\left(\forall a, P'.\ P \xrightarrow{a} P' \implies \exists Q'.\ Q \xrightarrow{a} Q' \wedge P'\,\mathcal{S}\,Q'\right)$$

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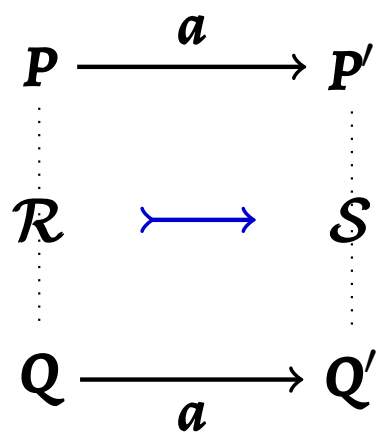

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Bisimilarity ($\sim \subseteq \text{Proc} \times \text{Proc}$) is:

- the union of all bisimulations, or, equivalently
- the greatest fixed point of $\rightarrowtail$.

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The bisimulation proof method is well suited to exhaustive search using automated techniques.

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To show P and Q are similar,

1. **Initialize the *simulation set* to empty**
2. **If (P, Q) is in the simulation set, succeed.**
   Else, add (P, Q) to the simulation set
3. **For every P' with $P \xrightarrow{a} P'$,**
   a. **Find a Q' with $Q \xrightarrow{a} Q'$** **(failure, backtracking)**
   b. **Recursively-goto step 2 with P' and Q'**

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Abella

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- Simply typed intuitionistic logic
- Least and greatest *relational* fixed point definitions
- An interactive tactics-based proof environment

- Quantification over λ-terms and higher-order syntax
- Equality of λ-terms upto $\alpha\beta\eta$
- Binding and scope issues reduced to quantifier alternation instead of side conditions
- Generic quantification

- Built-in executable *specification logic* (subset of λProlog)

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In Abella

$$P \sim Q \triangleq_{\nu}$$
$$\left(\forall a, P'.\ P \xrightarrow{a} P' \implies \exists Q'.\ Q \xrightarrow{a} Q' \wedge P' \sim Q'\right) \wedge$$
$$\left(\forall a, Q'.\ Q \xrightarrow{a} Q' \implies \exists P'.\ P \xrightarrow{a} P' \wedge P' \sim Q'\right)$$

In Abella

$$P \sim Q \triangleq_{\nu} \left(\forall a, P'.\ P \xrightarrow{a} P' \implies \exists Q'.\ Q \xrightarrow{a} Q' \wedge P' \sim Q'\right) \wedge \left(\forall a, Q'.\ Q \xrightarrow{a} Q' \implies \exists P'.\ P \xrightarrow{a} P' \wedge P' \sim Q'\right)$$

```
CoDefine bisim : proc → proc → prop by
  bisim P Q ≜
     (forall A P', one P A P' →
        exists Q', one Q A Q' ∧ bisim P' Q')
   ∧ (forall A Q', one Q A Q' →
        exists P', one P A P' ∧ bisim P' Q').
```

In Abella

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```
Define one : proc → action → proc → prop by
  ...
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```
Kind proc, action     type.
...
```

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```

In Abella: CCS (signature)

```
Kind name,action,proc type.

% Action constructors
Type tau          action.
Type up,dn        name → action.

% Process constructors
Type null         proc.
Type plus,par     proc   → proc → proc.
Type act          action → proc → proc.
Type repl         proc   → proc.

% Dual actions
Define dual : action → action → prop by
  dual (up A) (dn A)
; dual (dn A) (up A).
```

Digression: Definitions by Pattern Matching

```
Define dual : action → action → prop by
  dual (up X) (dn X)
; dual (dn X) (up X).

Define dual' : action → action → prop by
  dual A B ≜
    (exists X, A = up X ∧ B = dn X) ∨
    (exists X, A = dn X ∧ B = up X).
```

In Abella: CCS (steps)

```
Define one : proc → action → proc → prop by
  one (act A P) A P
; one (plus P1 P2) A Q              ≜ one P1 A Q
; one (plus P1 P2) A Q              ≜ one P2 A Q
; one (par P Q) A (par P1 Q)        ≜ one P A P1
; one (par P Q) A (par P Q1)        ≜ one Q A Q1
; one (repl P) A (par (repl P) Q)   ≜ one P A Q
; one (par P Q) tau (par P1 Q1)     ≜
    exists A B, dual A B ∧ one P A P1 ∧ one Q B Q1
; one (repl P) tau (par (repl P) (par Q R)) ≜
    exists A B, dual A B ∧ one P A Q ∧ one P B R.
```

Example Meta-Theorem: Bisimilarity is a Congruence

```
Theorem bisim_plus1_cong :
  forall P Q R,
    bisim P Q → bisim (plus P R) (plus Q R).
```

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```
Theorem bisim_plus1_cong :
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coinduction.
intros PbiQ.    % PbiQ : bisim P Q
case PbiQ. split.
  intros PaP1. % PaP1 : one P A P1
  case PaP1.
    ...

  intros QaQ1. % QaQ1 : one Q A Q1
    ...
```

Bisimulation Up To

Infinite Bisimulation Sets

$$!(a + b) \sim !a \mid !b$$

Infinite Bisimulation Sets

$$!(a + b) \sim !a \mid !b$$

$$!(a + b) \xrightarrow{a} !(a + b) \mid 0 \xrightarrow{b} !(a + b) \mid 0 \mid 0$$

Infinite Bisimulation Sets

$$!(a + b) \sim !a \mid !b$$

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$$!a \mid !b \xrightarrow{a} (!a \mid 0) \mid !b \xrightarrow{b} (!a \mid 0) \mid (!b \mid 0)$$

Infinite Bisimulation Sets

$$\boxed{!(a+b) \sim\ !a \mid !b}$$

$$!(a+b) \xrightarrow{a} !(a+b) \mid 0 \xrightarrow{b} !(a+b) \mid 0 \mid 0$$

$$!a \mid !b \xrightarrow{a} (!a \mid 0) \mid !b \xrightarrow{b} (!a \mid 0) \mid (!b \mid 0)$$

In this case we have a closed form "generalization"

$$!(a+b) \mid 0^{m+n} \quad \sim \quad (!a \mid 0^{m}) \mid (!b \mid 0^{n})$$

Infinite Bisimulation Sets

$$!!a \sim !a$$

$$!!a \xrightarrow{a} !!a \mid (!a \mid 0) \xrightarrow{a} !!a \mid (!a \mid 0) \mid (!a \mid 0) \xrightarrow{a} !!a \mid (!a \mid 0 \mid 0) \mid (!a \mid 0)$$

$$!a \xrightarrow{a} !a \mid 0 \xrightarrow{a} !a \mid 0 \mid 0 \xrightarrow{a} !a \mid 0 \mid 0 \mid 0 \mid 0$$

Infinite Bisimulation Sets

$$!!a \sim !a$$

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$$!a \xrightarrow{a} !a \mid 0 \xrightarrow{a} !a \mid 0 \mid 0 \xrightarrow{a} !a \mid 0 \mid 0 \mid 0 \mid 0$$

Closed form for the generalization is difficult!

Bisimulation Up To

[Milner, Sangiorgi, Pous, ...]

Simple idea:

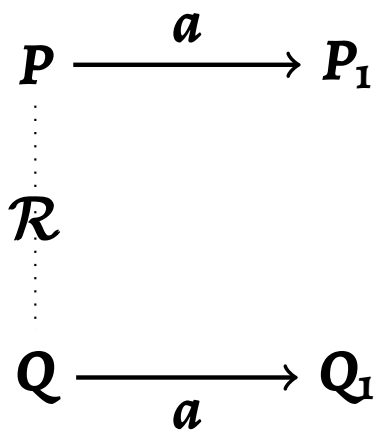
> ***Before making the recursive call to the procedure, modify the process pair to a more reasonable form.***

Example:

$$!!a \xrightarrow{a} !!a \mid (!a \mid 0) \rightsquigarrow !!a$$

$$!a \xrightarrow{a} !a \mid 0 \rightsquigarrow !a$$

Progress Up To



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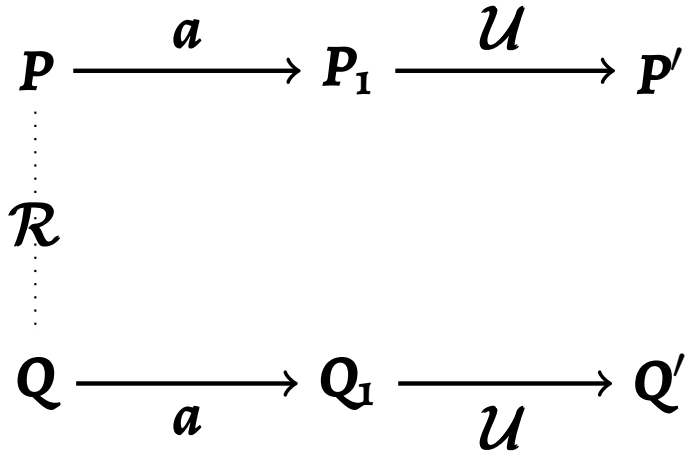

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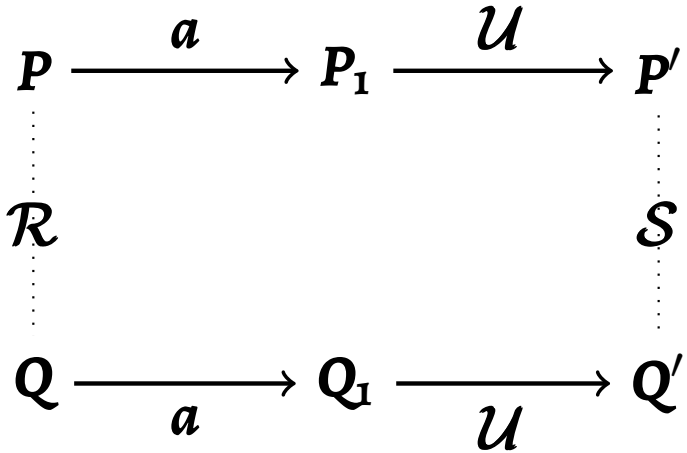

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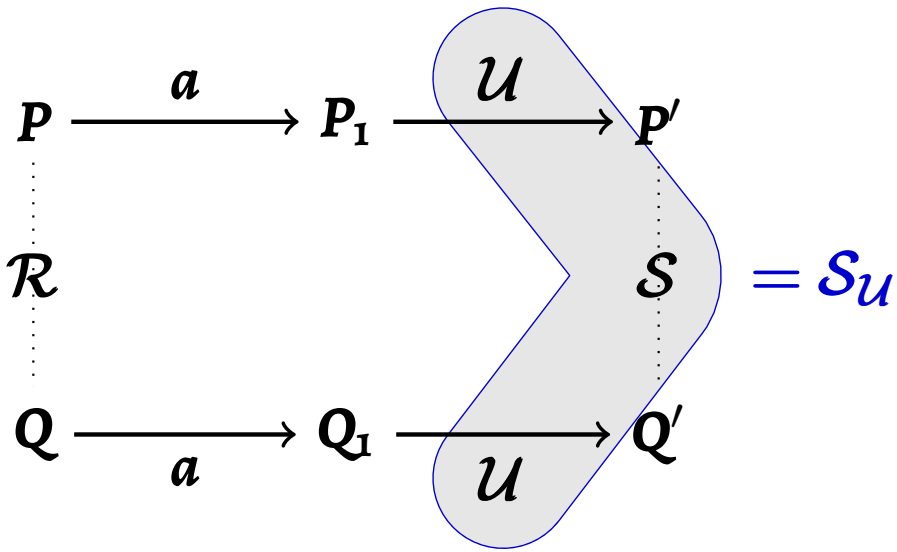

Progress Up To

$$(\mathcal{U} \subseteq \mathbf{Proc} \times \mathbf{Proc} \times \mathbf{Proc} \times \mathbf{Proc})$$

$$P \,\mathcal{R}_{\mathcal{U}}\, Q \triangleq \exists P', Q'.\ \mathcal{U}(P, P', Q, Q') \wedge P' \,\mathcal{R}\, Q'$$

$$\begin{aligned}
&\mathcal{R} \rightarrowtail_{\mathcal{U}} \mathcal{S} \triangleq \\
&\quad \forall P, Q.\ P \,\mathcal{R}\, Q \implies \\
&\qquad \left(\forall a, P'.\ P \xrightarrow{a} P' \implies \right. \\
&\qquad\qquad \left. \exists Q'.\ Q \xrightarrow{a} Q' \wedge P' \,\mathcal{S}_{\mathcal{U}}\, Q'\right) \wedge \\
&\qquad \left(\forall a, Q'.\ Q \xrightarrow{a} Q' \implies \right. \\
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\end{aligned}$$

Bisimulation and Bisimilarity Up To

Given a $\mathcal{U} \subseteq \text{Proc} \times \text{Proc} \times \text{Proc} \times \text{Proc}$,

$\mathcal{R} \subseteq \text{Proc} \times \text{Proc}$ **is a bisimulation up to** $\mathcal{U}$ **if** $\mathcal{R} \rightarrowtail_{\mathcal{U}} \mathcal{R}$.

Bisimilarity up to $\mathcal{U}$ ($\sim_{\mathcal{U}} \subseteq \text{Proc} \times \text{Proc}$) **is:**

- **the union of all bisimulations up to** $\mathcal{U}$**, or, equivalently**
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Note: $\sim \; = \; \sim_{\mathrm{Id}}$ **where:**

$$\mathrm{Id}(P, P', Q, Q') \triangleq P' = P \wedge Q' = Q$$

Up To Techniques in Abella

```
Define upto : (proc → proc → proc → proc → prop)
           → (proc → proc → prop)
           → (proc → proc → prop) by
  upto UT Rel P Q ≜
    exists P' Q', UT P P' Q Q' ∧ Rel P' Q'.

Define bisupto : (proc → proc → proc → proc → prop)
              → (proc → proc → prop) by
  bisupto UT P Q ≜
     (forall A P', one P A P' →
        exists Q', one Q A Q' ∧ upto UT (bisupto UT) P' Q')
   ∧ (forall A Q', one Q A Q' →
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     (forall A P1, one P A P1 →
        exists Q1, one Q A Q1 ∧
          exists P2 Q2, UT P1 P2 Q1 Q2 ∧ bisupto UT P2 Q2)
   ∧ (forall A Q1, one Q A Q1 →
        exists P1, one P A P1 ∧
          exists P2 Q2, UT P1 P2 Q1 Q2 ∧ bisupto UT P2 Q2).
```

Soundness

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An up to technique $\mathcal{U}$ is sound if

- **the fixed points of $\rightarrowtail_{\mathcal{U}}$ are contained in the fixed points of $\rightarrowtail$, or, equivalently**
- **if $\sim_{\mathcal{U}} \subseteq \sim$.**

Soundness in Abella

```
Define sound : (proc → proc → proc → proc → prop)
             → prop by
  sound UT ≜
    forall P Q, bisupto UT P Q → bisim P Q.
```

Up To Techniques

for CCS

Up To Bisimilarity

The up to bisimilarity technique $\mathcal{B}$ is given by:

$$\mathcal{B}(P, P', Q, Q') \triangleq P \sim P' \wedge Q \sim Q'$$

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Let $\mathcal{R}$ be a bisimulation up to bisimilarity. Then:

$$\begin{array}{ccc} P & \xrightarrow{a} P_1 & \sim & P' \\ \mathcal{R} & & & \mathcal{R} \\ Q & \xrightarrow[a]{} Q_1 & \sim & Q' \end{array}$$

Up To Bisimilarity

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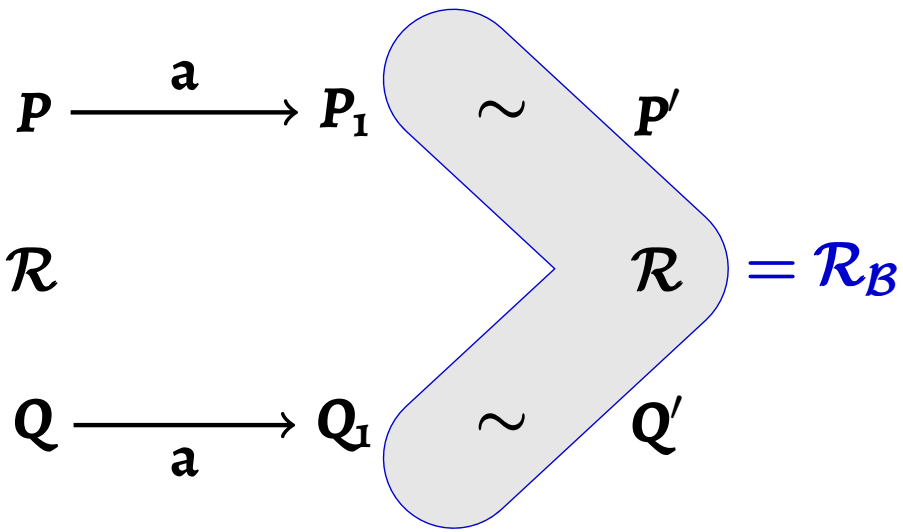

Up To Bisimilarity

```
Define bisim_t : proc → proc → proc → proc → prop by
  bisim_t P1 P2 Q1 Q2 ≜ bisim P1 P2 ∧ bisim Q1 Q2.
```

Can show (in Abella):

- $!!a \sim_{\mathcal{B}} !a$
- $!(a + b) \sim_{\mathcal{B}} !a \mid !b$

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```
Theorem example_2 : forall A B,
  bisupto bisim_t (repl (plus (act (dn A) null) (act (dn B) null)))
                  (par (repl (act (dn A) null)) (repl (act (dn B) null))).
coinduction. intros. unfold. split.
  % ...
```

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coinduction. intros. unfold. split.
  % ...
```

Need `sound bisim_t` to conclude that they are `bisim`!

Soundness of Up To Bisimilarity

```
Define sound : (proc → proc → proc → proc → prop) → prop by
  sound UT ≜ forall P Q, bisupto UT P Q → bisim P Q.

Theorem upto_bisim_sound : sound bisim_t.
```

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Lemma 6.2.3 *The up-to-bisimilarity technique is sound; that is, if $\mathcal{R}$ is a bisimulation up to bisimilarity, then $\mathcal{R} \subseteq \sim$.*

Proof If $\mathcal{R}$ is a bisimulation up to bisimilarity then $\sim\mathcal{R}\sim$ (the composition of the three relations) is a bisimulation and $\sim\mathcal{R}\sim \subseteq \sim$. Since $\sim$ contains the identity relation, we also have $\mathcal{R} \subseteq \sim\mathcal{R}\sim$ which allows us to conclude. □

[Pous and Sangiorgi]

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```
Theorem bisim_t_right : forall P Q,
  (exists P1 Q1, bisim P P1 ∧ bisupto bisim_t P1 Q1
                 ∧ bisim Q1 Q) →
    bisim P Q.

Theorem bisim_t_left : forall P Q,
  bisupto bisim_t P Q →
    (exists P1 Q1, bisim P P1 ∧ bisupto bisim_t P1 Q1
                   ∧ bisim Q1 Q).
```

Soundness of Up To Bisimilarity: Proof

```
Theorem bisim_t_right : forall P Q,
  (exists P1 Q1, bisim P P1 ∧ bisupto bisim_t P1 Q1
                 ∧ bisim Q1 Q) →
    bisim P Q.
coinduction. intros. unfold.
   % case analysis

Theorem bisim_t_left : forall P Q,
  bisupto bisim_t P Q →
    (exists P1 Q1, bisim P P1 ∧ bisupto bisim_t P1 Q1
                   ∧ bisim Q1 Q).
% by reflexivity and symmetry of bisim
%   (each proved by a straightforward co-induction)
```

Soundness of Up To Bisimilarity: An Alternative

[Cimini]

$\mathcal{B}$ **is sound if and only if** $\sim_{\mathcal{B}}$ **is an equivalence relation.**

Up To Context

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The up to context technique $\mathcal{K}$ is given by:

$$\mathcal{K}(P, P', Q, Q') \triangleq \exists C.\, P = C[P'] \wedge Q = C[Q']$$

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If $\mathcal{R}$ is a bisimulation up to context, then:

$$\begin{array}{ccccc} P & \xrightarrow{\mathbf{a}} & C[P'] & \xleftarrow{C[-]} & P' \\ \mathcal{R} & & & & \mathcal{R} \\ Q & \xrightarrow[\mathbf{a}]{} & C[Q'] & \xleftarrow[C[-]]{} & Q' \end{array}$$

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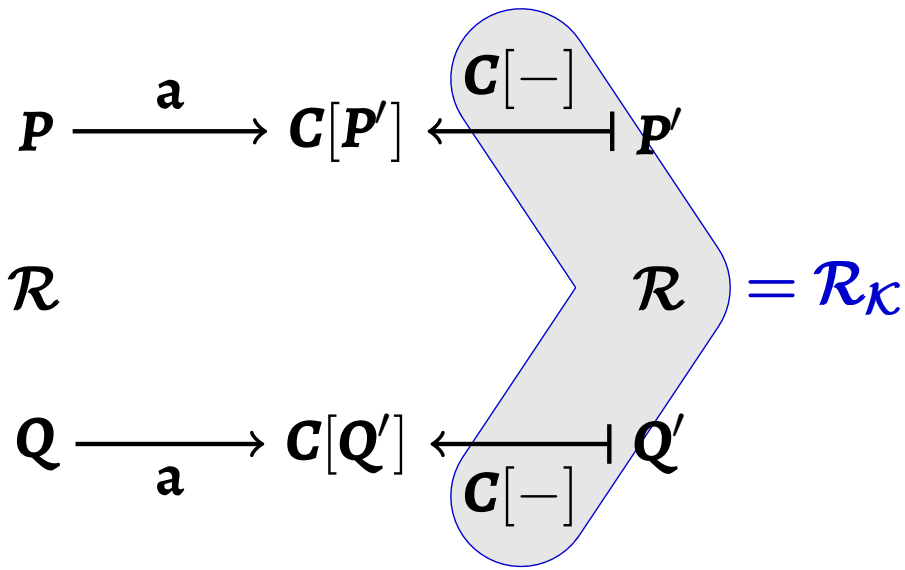

Up To Bisimilarity And Context

The up to bisimilarity and context technique $\mathcal{BK}$ is given by:

$$\mathcal{BK}(P, P', Q, Q') \triangleq \exists C.\, P \sim C[P'] \wedge Q \sim C[Q']$$

Up To Bisimilarity And Context

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If $\mathcal{R}$ is a bisimulation up to bisimilarity and context, then:

$$\begin{array}{ccccc} P & \xrightarrow{a} & P_1 & \sim & C[P'] \xleftarrow{C[-]} P' \\ \mathcal{R} & & & & \mathcal{R} \\ Q & \xrightarrow[a]{} & Q_1 & \sim & C[Q'] \xleftarrow[C[-]]{} Q' \end{array}$$

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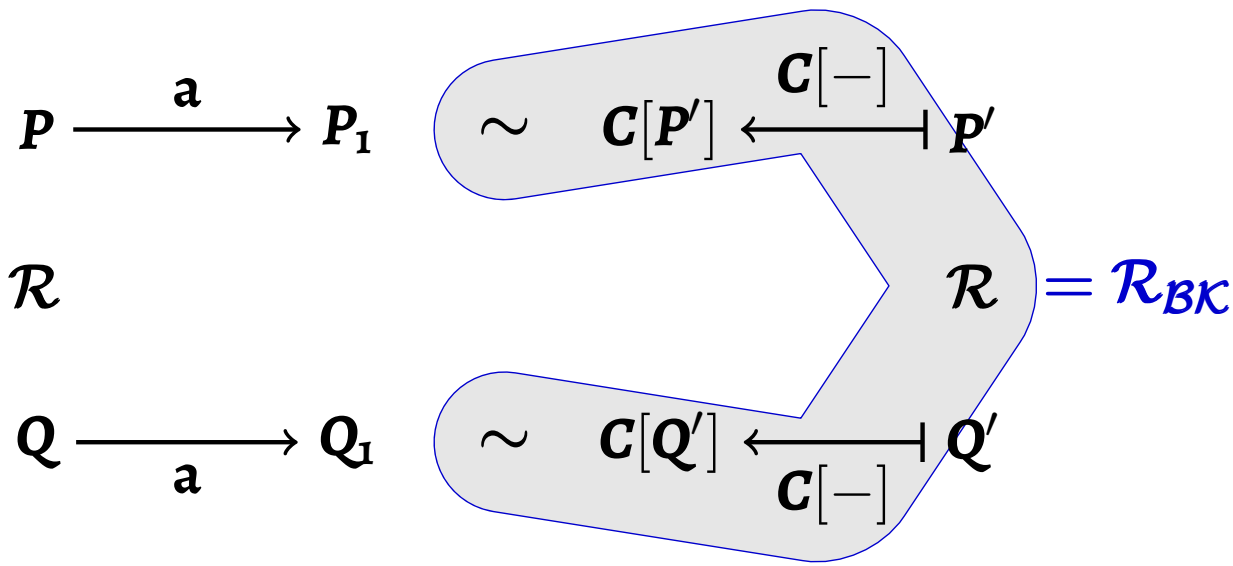


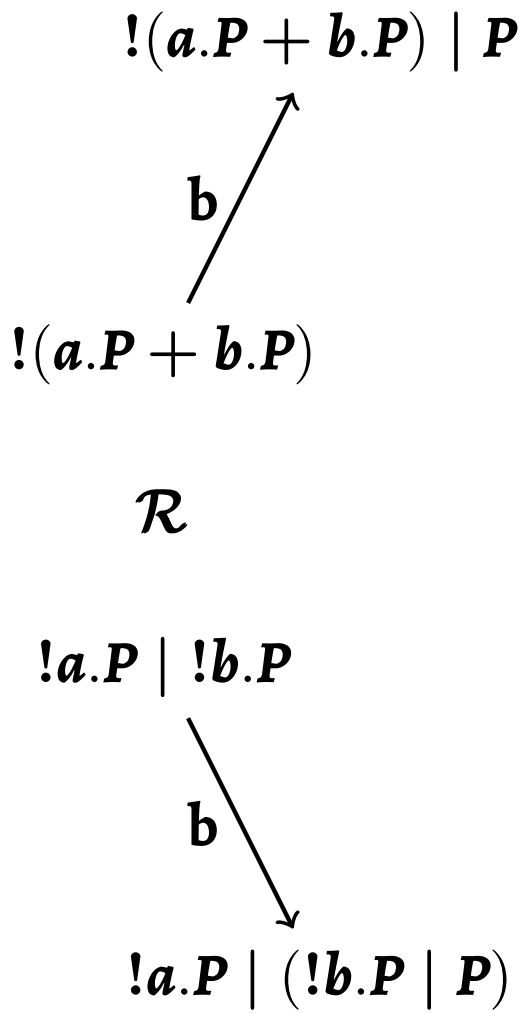
Example

$$!(a.P + b.P)$$

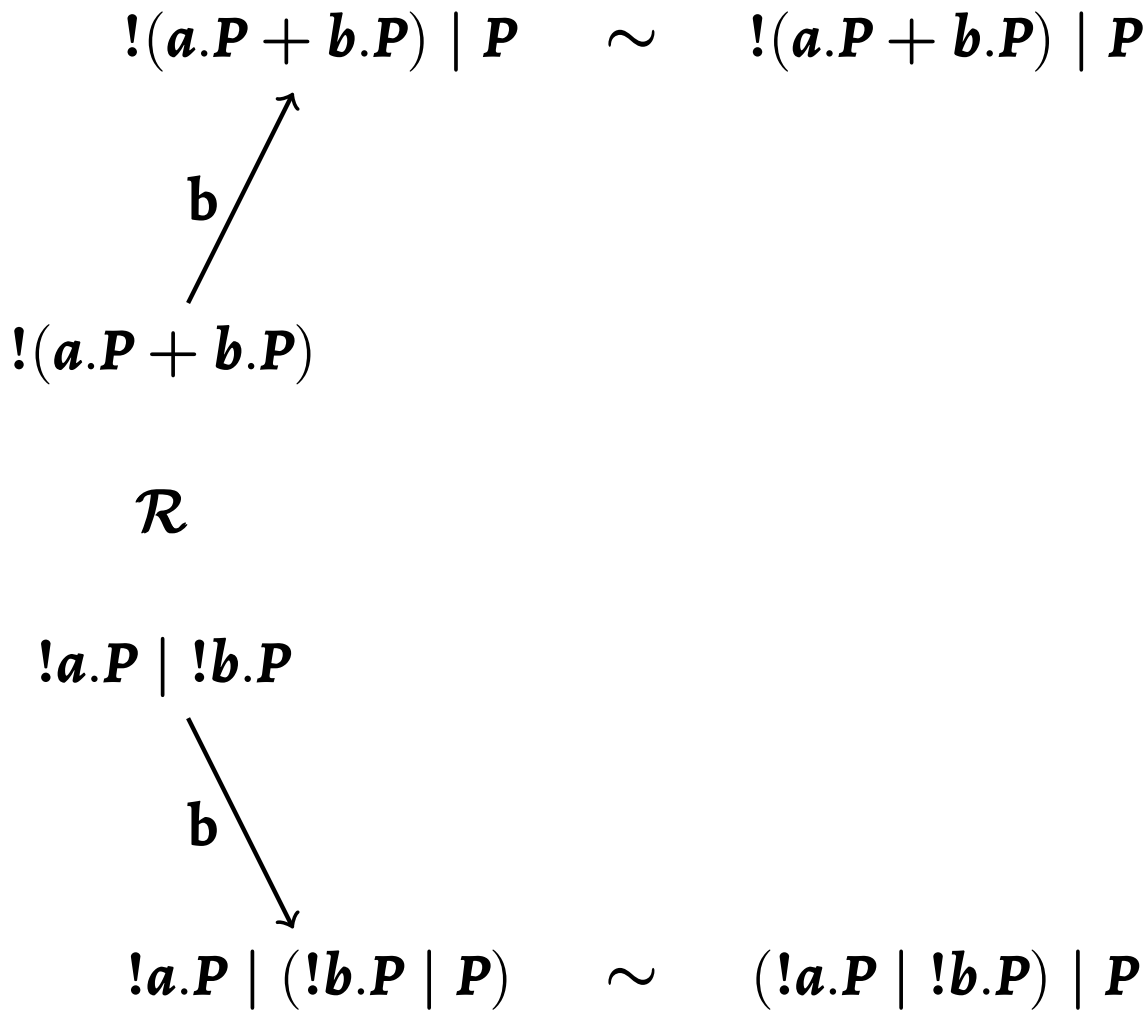
$$\mathcal{R}$$

$$!a.P \mid !b.P$$

Example



Example



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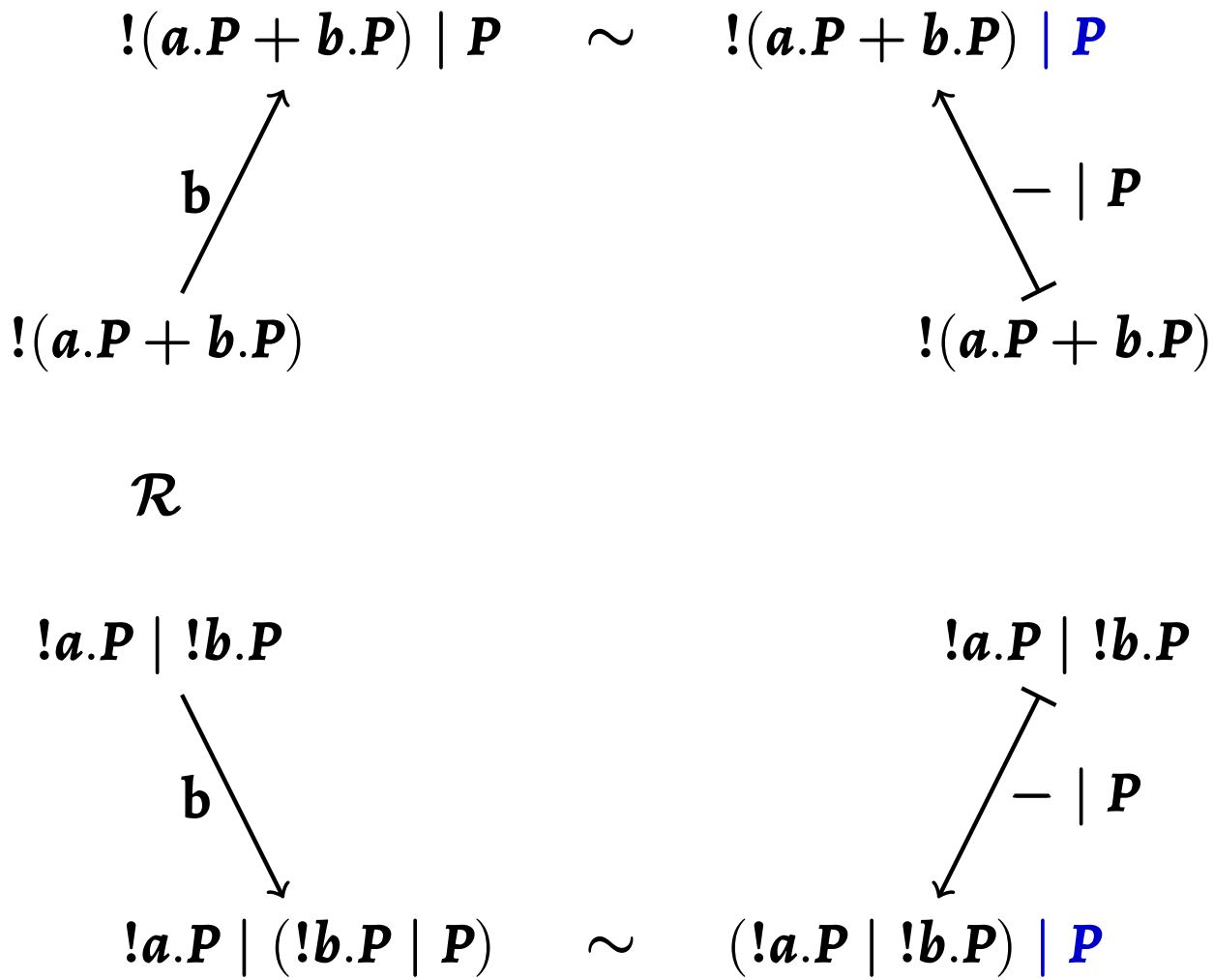

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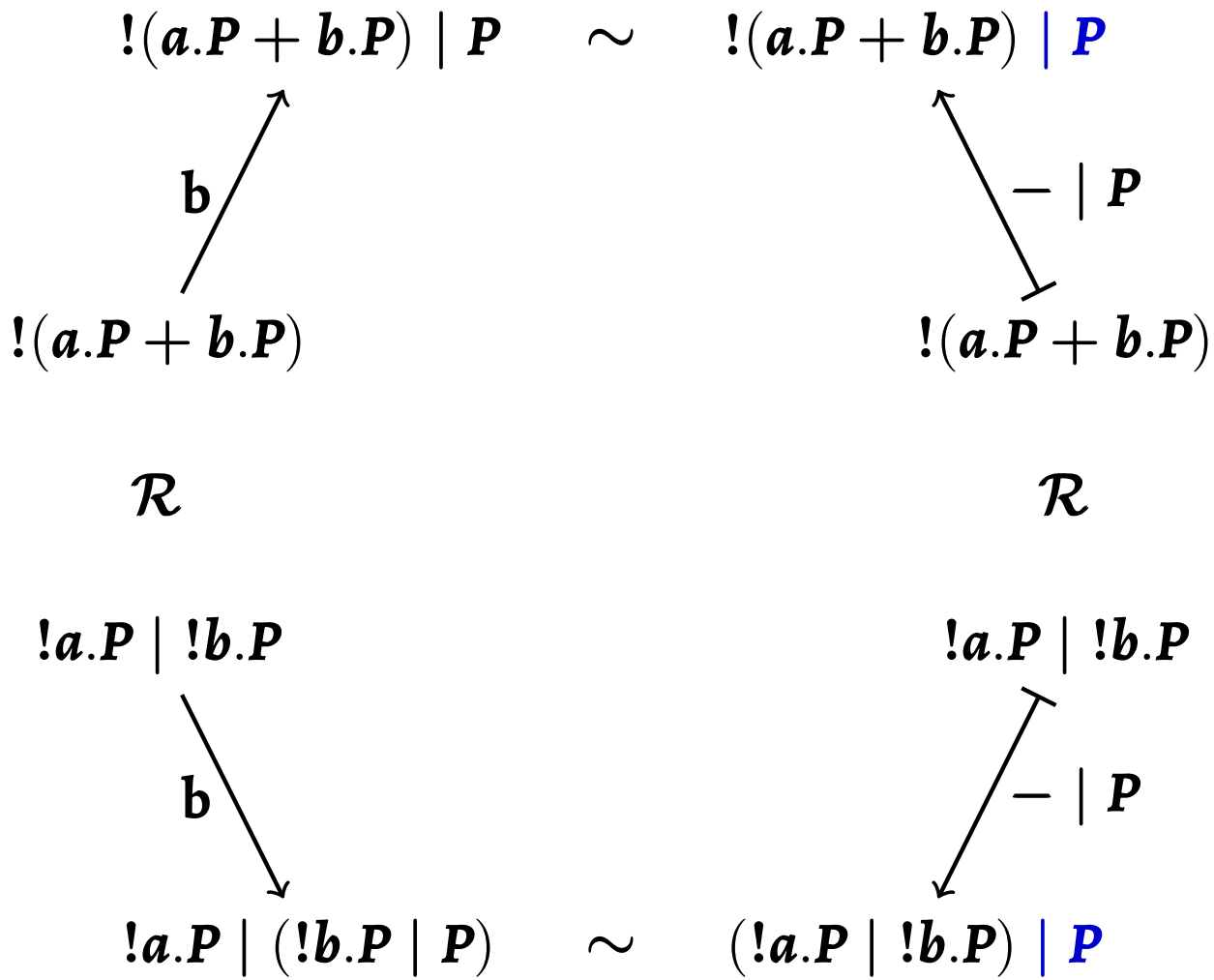

$\mathcal{BK}$ in Abella

```
Define k_t : proc → proc → proc → proc → prop by
  k_t P1 P2 Q1 Q2 ≜ exists C,
    in_ctx C P1 P2 ∧ in_ctx C Q1 Q2.

Define bk_t : proc → proc → proc → proc → prop by
  bk_t P1 P2 Q1 Q2 ≜ exists P3 Q3,
    bisim_t P1 P3 Q1 Q3 ∧ k_t P3 P2 Q3 Q2.
```

Soundness of Up To $\mathcal{BK}$

[Cimini]

$\mathcal{BK}$ **is sound if** $\sim_{\mathcal{BK}}$ **is a congruence.**

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Proof: by straightforward co-induction.

Congruence of $\sim_{\mathcal{BK}}$

[Cimini]

$\sim_{\mathcal{BK}}$ **is a congruence if contexts are faithful**

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[Cimini]

$\sim_{\mathcal{BK}}$ **is a congruence if contexts are faithful**

Proof: by straightforward *induction* on the structure of contexts.

Faithful Contexts

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Fairly well known result, but now formalized in Abella.

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Fairly well known result, but now formalized in Abella.

Laborious, but doable, proof.

Sketch of Soundness of $\mathcal{BK}$

CCS contexts are faithful

$\sim_{\mathcal{BK}}$ is a congruence

$\mathcal{BK}$ is sound

Ongoing Work

Several other up to techniques in the literature

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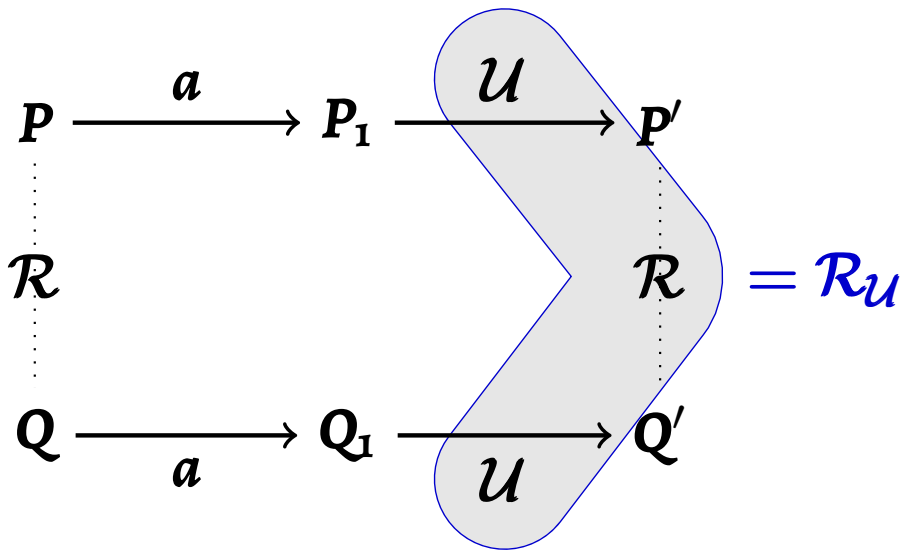
Examples:

- **Transitivity:** $\mathcal{R} \rightarrowtail \mathcal{R}^*$
- **Union Bisimilarity:** $\mathcal{R} \rightarrowtail \mathcal{R} \cup \sim$

Ongoing Work: Up To Transitivity

Problem: Is there a $\mathcal{U}$ such that $\mathcal{R}_{\mathcal{U}} = \mathcal{R}^*$?



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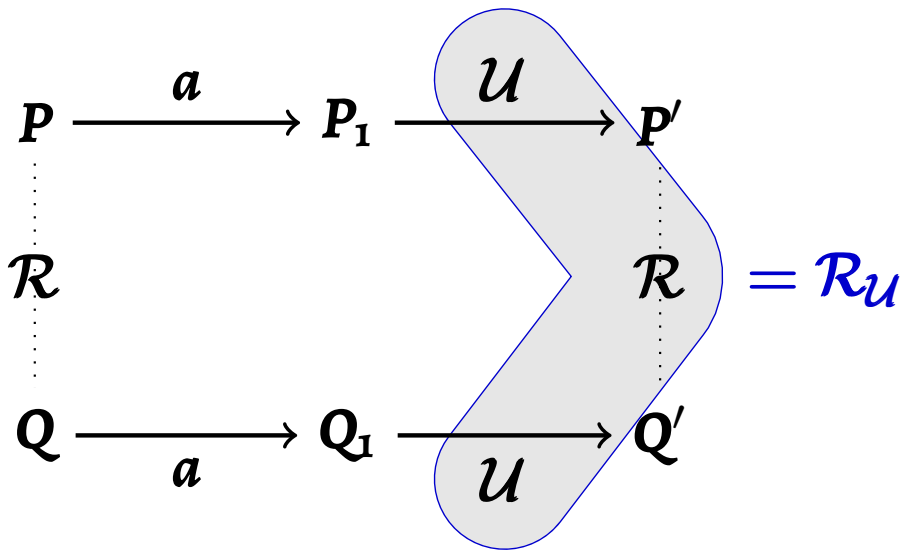

Almost! $\quad \mathcal{U}(P, P', Q, Q') \triangleq P \mathcal{R}^* P' \wedge Q \mathcal{R}^* Q'.$

Bisimulation Up To for the π-Calculus

Higher-Order Encoding

```
Type null          proc.
Type taup          proc → proc.
Type plus, par     proc → proc → proc.
Type match, out    name → name → proc → proc.
Type repl          proc → proc.

Type in            name → (name → proc) → proc.
Type nu            (name → proc) → proc.
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```

$(\nu x)\ a(u).\ \overline{u}\langle x\rangle.\ 0$ $\qquad$ $\texttt{nu}\ (\lambda x.\ \texttt{in}\ a\ (\lambda u.\ \texttt{out}\ u\ x\ \texttt{null}))$

Digression: Representational Adequacy

Higher-Order Abstract Syntax; λ-Tree Syntax

π-calculus process $\cong$ term of type `proc`

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π-calculus process $\cong$ term of type `proc`

- The only term of type $A \to B$ is of the form $\lambda(x{:}A).\, M$
- Therefore, can access the "body" of a λ-term (by higher-order unification)
- Substitution is by application and normalization
- There are no "exotic terms", *i.e.*, terms of type `proc` that do not denote a π-calculus process

Bound Actions

```
Define
 one  : proc → action           → proc           → prop,
 oneb : proc → (name → action) → (name → proc) → prop
by
   ... % cases of tau, plus, par, repl, symmetry
; one (out X Y P) (up X Y) P
; one (nu λx. P x) A (nu λx. Q x) ≜
    nabla x, one  (P x) A (Q x)
; one (par P Q) tau (nu λy. par (M y) (N y)) ≜
    exists X, oneb P (dn X) M ∧ oneb Q (up X) N

; oneb (in X M) (dn X) M
; oneb (nu λx. P x) A (λy. nu λx. Q x y) ≜
    nabla x, oneb (P x) A (λy.Q x y)
; oneb (nu λx. M x) (up X) N ≜
    nabla y, one (M y) (up X y) (N y)
; ... % cases of plus, par, repl
```

Bisimilarity for the π-Calculus

```
CoDefine bisim : proc → proc → prop by
  bisim P Q ≜
    (∀A P1, one P A P1 → ∃Q1, one Q A Q1 ∧ bisim P1 Q1)
∧   (∀X M, oneb P (dn X) M → ∃N, oneb Q (dn X) N ∧
      (∀w, bisim (M w) (N w)) )
∧   (∀X M, oneb P (up X) M → ∃N, oneb Q (up X) N ∧
       nabla w, bisim (M w) (N w))
∧   (∀A Q1, one Q A Q1 → ∃P1, one P A P1 ∧ bisim Q1 P1)
∧   (∀X N, oneb Q (dn X) N → ∃M, oneb P (dn X) M ∧
      (∀w, bisim (N w) (M w)))
∧   (∀X N, oneb Q (up X) N → ∃M,  oneb P (up X) M ∧
       nabla w, bisim (N w) (M w)).
```

Bisimilarity Up To for the π-Calculus

```
CoDefine bisupto : (proc → proc → proc → proc → prop)
                 → (proc → proc → prop) by
  bisupto UT P Q ≜
   (∀A P1, one P A P1 → ∃Q1, one Q A Q1 ∧
      upto UT (bisupto UT) P1 Q1)
∧  (∀X M, oneb P (dn X) M → ∃N, oneb Q (dn X) N ∧
     (∀w, upto UT (bisupto UT) (M w) (N w)) )
∧  (∀X M, oneb P (up X) M → ∃N, oneb Q (up X) N ∧
     nabla w, upto UT (bisupto UT) (M w) (N w))
∧  (∀A Q1, one Q A Q1 → ∃P1, one P A P1 ∧
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∧  (∀X N, oneb Q (dn X) N → ∃M, oneb P (dn X) M ∧
     (∀w, upto UT (bisupto UT) (N w) (M w)))
∧  (∀X N, oneb Q (up X) N → ∃M,  oneb P (up X) M ∧
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```

Modularity in Definitions

We can reuse the definitions already seen earlier.

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```
CoDefine
 bisupto_bisim : proc → proc → prop,
 bisupto_k     : proc → proc → prop,
 bisupto_bk    : proc → proc → prop
by
  bisupto_bisim P Q ≜ bisupto bisim_t P Q
; bisupto_k P Q     ≜ bisupto k_t P Q
; bisupto_bk P Q    ≜ bisupto bk_t P Q.la
```

Status of Meta-Proofs for the π-Calculus

- **Soundness of $\mathcal{B}$: DONE**
  - **Factoring: DONE**
  - **Characterization as equivalence of $\sim_{\mathcal{B}}$: DONE**
- **Soundness of $\mathcal{BK}$: in progress**
  - **Generalization of faithful contexts for the π-calculus: DONE**
  - **Faithful contexts implies Congruence of $\sim_{\mathcal{BK}}$: in progress**
  - **Congruence of $\sim_{\mathcal{BK}}$ implies soundness: DONE**

Perspectives

Related Work: Compatibility

[Sangiorgi, Pous, …]

- When are two up to techniques composable?
- Mathematics has been thoroughly worked out
- Soundness of a technique is a corollary of compatibility
- Formalized in Coq by Pous
  - Builds up from first principles:
    — lattices, functions, fixed points, *etc.*
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- In Abella?

Related Work: Co-Induction Modulo Rewriting in Coq

- **Paco: A Coq Library for Parameterized Coinduction'**
  [Hur, Neis, Dreyer, Vafeiadis]

  > *The Paco library provides a tactic called "**pcofix**", replacing Coq's primitive **cofix** and avoiding its syntactic guardedness checking of proof terms. We have found that **pcofix** yields clear [...] usability benefits, even on simple examples.*

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  - **Less clear for the π-calculus**

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(Higher-order syntax makes reasoning about binding structures and bound actions systematic and natural in Abella, while this is problematic for first-order syntax or rich function types.)